Dominance-Inspired Incomparability for Function-Based Ranking Methods

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Abstract

A popular task within Multi-Criteria Decision Aid (MCDA) is the ranking problem, where alternatives are ranked from the most preferred to the least preferred. There are two standard paradigms tackling such problems: a functional one that naturally admits indifference of alternatives or preference of one alternative over another, and a relational one that additionally admits incomparability. Those paradigms also differ in the amount of preferential information obtained from a decision maker, which is much larger for the relational model. Inspired by the commonly accepted objective dominance relation, in this paper, we theorize on the possibilities of incorporating incomparability into the functional paradigm. As a result, the extended family of relations will admit incomparability alongside indifference and preference, require as little preferential information as possible, and have the undisputed objective preferential relation as its foundation.

1 Introduction

Multi-Criteria Decision Aid (MCDA) is extensively used across diverse industries to support decision makers (DMs) in solving problems that involve real-world objects (alternatives) evaluated on multiple conflicting criteria. Often, this entails ranking alternatives from the most preferred to the least preferred. Ranking methods are commonly applied in high-stake domains that impact societies like sustainable energy planning [Becchio *et al.*, 2021], manufacturing [Wang, 2009; Zhang *et al.*, 2023], marketing [Yu *et al.*, 2011], sustainable development [Piwowarski *et al.*, 2018], and engineering [Lin *et al.*, 2023]. For an extended overview of MCDA methods, models, and frameworks, see reviews [Belton and Stewart, 2002; Ishizaka and Nemery, 2013; Bisdorff *et al.*, 2015; Greco *et al.*, 2016; Cinelli *et al.*, 2022].

Among methods that tackle the ranking problem, two standard, predominantly popular paradigms emerge: a *functional paradigm* (represented by, e.g., TOPSIS and SAW [Hwang and Yoon, 1981], the UTA family of methods [Jacquet-Lagrèze and Siskos, 1982]) and a *relational paradigm* (represented by, e.g., the ELECTRE family of methods [Roy, 1991] or PROMETHEE I/II [Brans, 1982].

Functional approaches produce non-negative real values for individual alternatives, which determine a total pre-order that can be used for ranking. For example, TOPSIS calculates distances between the considered alternatives and two predefined ones, namely the ideal and the anti-ideal, and creates a ranking of the alternatives according to a chosen aggregation of these distances. As a result, functional methods naturally reflect indifference and preference between pairs of alternatives. In the first case, two alternatives are seen as equally good and, as such, occupy the same ranking position. In case of preference, one alternative is regarded as being better than the other one and is thus placed higher in the ranking. The functional paradigm, however, does not have the potential to model *incomparability* between two alternatives, i.e., the case where one alternative is better than the other on some criteria but worse on other criteria. This is unfortunate, as incomparability often comes as a natural interpretation of the relation between alternatives. As an example, consider a case of two students being characterized by their marks in two equally important courses. Let us suppose that Student A scored 100% in the first course and 0% in the second, while Student B scored 0% in the first course and 100% in the second. Because Student A is so much better on the first criterion and so much worse on the second than Student B, in such a scenario, it is incomparability that would best express the relation between those students. Unfortunately, methods implementing the functional paradigm would invariably treat these alternatives as indifferent (e.g., in TOPSIS, the aggregation of the distances would be exactly the same).

The relational paradigm, on the other hand, generates real values for pairs of alternatives, thus modeling an auxiliary, valued relation. For example, ELECTRE methods define an outranking relation in this role. To arrive at the relations of indifference, preference, or incomparability, the outranking values are scrutinized in a dedicated way. If outranking holds for a pair of alternatives in both ways, the two alternatives enter the relation of indifference. If it holds in one way but not the other, the two alternatives enter the relation of preference. Otherwise, i.e., when it does not hold in any way, the two alternatives enter the relation of incomparability. This paradigm is thus, by definition, capable of expressing incomparability, which would constitute the most probable result for the discussed Student A and Student B. Admission of incomparability is definitely a strong point of relational methods, but it comes at the cost of a large amount of preferential information that needs to be delivered by the DM. For example, to define the outranking, the ELECTRE methods require the following parameters for each criterion: a weight, an indifference threshold, a preference threshold, and a veto threshold. Requisition of those parameters also introduces some 'degrees of freedom', so that it may become uncertain whether the decisions about incomparability are due to the particular values of the parameters or to the particular features of the alternatives.

In this context, an interesting research question emerges: *Is it possible to extend the functional paradigm so that it would be capable of modeling explicit indifference, preference, and incomparability, but would require as little preferential information as possible?* In this paper, we put forward such an extension and theorize on its potential. Placing dominance relation as the inspiration for our approach allows to express indifference, preference, and incomparability on one hand, but also makes the paradigm not too demanding in terms of preferential information, as dominance is purely objective (in the sense that it requires no preferential information to be established).

2 Preliminaries

2.1 Alternatives and criteria

In MCDA, the real-world objects under consideration are commonly referred to as *alternatives*. In this paper, the set of all possible alternatives will be denoted by $\mathbb{A}$ and assumed to satisfy $\mathbb{A} \neq \varnothing$. The descriptions of alternatives consist of values of some pre-defined attributes. The counterdomain of each attribute is assumed to be a non-degenerated real-valued interval. The resulting, multi-dimensional descriptions are usually represented as vectors. In this paper, it is consistently assumed that *all* attributes are criteria (i.e. their domains are ordered according to preference in a weakly monotonic fashion), the set of which will be denoted by $\mathbb{K}$ and assumed to satisfy $\mathbb{K} \neq \varnothing$. Because the domain of every criterion $\mathcal{K} \in \mathbb{K}$ is a real-valued interval $\mathcal{V}$, it is bounded by two values: the least preferred (denoted by v_*) and the most preferred (denoted by v^*). Of course, the values of v_* and v^* may be different for different criteria. Additionally, according to the monotonicity type of the ordering of its domain, criteria may differ in their preference types. In particular, criterion $\mathcal{K} \in \mathbb{K}$:

- is referred to as of type 'gain' when its domain $\mathcal{V} = [v_*, v^*]$, and the preference of $v \in \mathcal{V}$ does not decrease with the increase of v,
- is referred to as of type 'cost' when its domain $\mathcal{V} = [v^*, v_*]$, and the preference of $v \in \mathcal{V}$ does not increase with the increase of v.

Distinction between 'gain' and 'cost' types constitutes preferential information delivered by the DM. However, MCDA methods often assume also other preference-related parameters. Depending on the amount of preferential information needed, we distinguish between 'preference-shallow' and 'preference-deep' contexts. They are defined informally as the totality of all assumptions that must be made if any preference-related conclusions (e.g., satisfiability of particular relations) regarding the underlying data are to be drawn. To get the intuition behind these notions, consider two alternatives, $a, b \in \mathbb{A}$, described in terms of a set of criteria. For example, to establish if a is *preferred over* b essentially requires running a method, which in turn, requires assuming all the criterion preference types and all the other preference-related information ('working parameters') that are necessary for the method to run. Notice that even when some values of the parameters are default and thus need not be explicitly specified, their existence must be assumed. As a result, the answer to the question of whether a is preferred over b is arrived at in a particular context of the assumed preferential information. We shall refer to this context as 'preference-deep'. On the other hand, to establish whether a is *dominated by* b requires assuming the criterion preference types only. We shall refer to this context as 'preference-shallow'.

The considerations contained in the following parts of this paper basically aim at finding ways to define convincing instances of incomparability between a and b in 'preference-shallow' contexts. As a result, we do not use weights in any further considerations, consistently retaining the 'preference-shallow' context of all analyses.

2.2 The utility space induced by criteria

Because simultaneous analyses of sets of criteria with differing intervals and differing types may be very inconvenient, we will use a simple transformation of a criterion by what will be referred to as its utility function $\mathcal{U} : \mathcal{V} \to [0, 1]$ [Susmaga *et al.*, 2023; Susmaga *et al.*, 2024]. Given:

- a domain $\mathcal{V} = [v_*, v^*]$ of a criterion $\mathcal{K} \in \mathbb{K}$ of type 'gain', the utility function $\mathcal{U}$ associated with $\mathcal{K}$ is defined as $\mathcal{U}(v) = \frac{v - v_*}{v^* - v_*}$ for $v \in \mathcal{V}$,
- a domain $\mathcal{V} = [v^*, v_*]$ of a criterion $\mathcal{K} \in \mathbb{K}$ of type 'cost', the utility function $\mathcal{U}$ associated with $\mathcal{K}$ is defined as $\mathcal{U}(v) = \frac{v_* - v}{v_* - v^*}$ for $v \in \mathcal{V}$.

Notice that the sole objective of the utility functions is to unify the intervals and the preference types of all criteria: while the original criteria may have different intervals and different types, the utility-transformed criteria will all have the same interval ($[0, 1]$) and the same type ('gain'). This means, in particular, that using the utility-transformed criteria simplifies the analyses without reducing their generality.

Given a set of criteria $\mathbb{K}$, $|\mathbb{K}| = n \geq 1$, consider the utility space US, i.e. the set of all possible vectors $[u_1, u_2, ..., u_n]$ such that $u_j \in [0, 1]$ for $j \in \{1, 2, ..., n\}$. The utility space US is thus an n-dimensional hypercube $[0, 1] \times [0, 1] \times ... \times [0, 1]$. Clearly, for each possible alternative, there exists $\mathbf{u} \in US$, which is the alternative's representation under the utility transformation. More precisely: if vector $E = [v_1, v_2, ..., v_n]$ is a representation of alternative e in terms of the criteria, then $[\mathcal{U}_1(v_1), \mathcal{U}_2(v_2), ..., \mathcal{U}_n(v_n)] \in US$, or (shortly) $\mathbf{u} = \mathcal{U}(E)$ is its representation in terms of the utilities. In particular, vectors $\mathbf{1} \in US$ and $\mathbf{0} \in US$ represent what is referred to as the ideal point (a hypothetical alternative with best possible descriptions: $[(v^*)_1, (v^*)_2, ..., (v^*)_n]$), and anti-ideal point (a hypothetical alternative with worst possible descriptions: $[(v_*)_1, (v_*)_2, ..., (v_*)_n]$), respectively.

2.3 Distances in utility space

Let $\mathbf{u} = [u_1, u_2, ..., u_n] \in US$, where n is the number of criteria. Then $sum(\mathbf{u})$ will be defined as $sum(\mathbf{u}) = \sum_{j=1}^{n} u_j$. Additionally, $mean(\mathbf{u})$ will be defined as $mean(\mathbf{u}) = \frac{sum(\mathbf{u})}{n}$. Notice that $sum(\mathbf{u}) \in [0, n]$, while $mean(\mathbf{u}) \in [0, 1]$ (which makes $mean(\mathbf{u})$ especially well interpretable). Finally, $\|\mathbf{u}\|$, the Euclidean norm (shortly: norm) will be defined as $\sqrt{\sum_{j=1}^{n} u_j^2}$. The norm is naturally interpreted as the length when applied to a vector and, as such, used to define the Euclidean distance (shortly: distance) between two points: $D(\mathbf{u}_a, \mathbf{u}_b) = \|\mathbf{u}_a - \mathbf{u}_b\|$.

Vectors $\mathbf{1}$ and $\mathbf{0}$ in US constitute the endpoints of the line segment that will be referred to as the main diagonal of US and denoted as $D_{\mathbf{0}}^{\mathbf{1}}$ (Figure 1). The length of $D_{\mathbf{0}}^{\mathbf{1}}$ is $\sqrt{n}$ and is the diagonal diameter (shortly: diameter) of US, while its half is the diagonal radius (shortly: radius) of US.

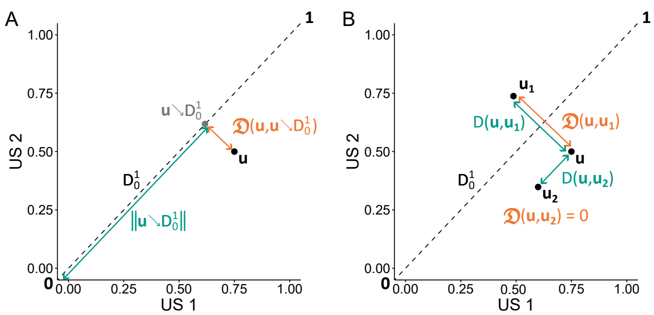


Figure 1: Distances in utility space. (A) US-based illustration of the orthogonal projection (grey point) of $\mathbf{u} = [0.75, 0.50]$ (black point) onto the diagonal $D_{\mathbf{0}}^{\mathbf{1}}$ (dashed line). Lengths of the coloured segments express the norm of $\mathbf{u} \searrow D_{\mathbf{0}}^{\mathbf{1}}$ (teal) and its orthogonal distance $\mathfrak{D}$ to $\mathbf{u}$ (orange). (B) Examples of Euclidean (teal) and orthogonal (orange) distances between $\mathbf{u}$ and two other points in US-space: $\mathbf{u_1} = [0.50, 0.75]$ and $\mathbf{u_2} = [0.60, 0.35]$.

The diagonal $D_{\mathbf{0}}^{\mathbf{1}}$ (dashed line in Figure 1A) is especially important because it determines what may be referred to as the multidimensional direction of additive preference. As such, it provides a geometric interpretation of the $mean(\mathbf{u})$ of any $\mathbf{u} \in US$. If one recalls orthogonal projections of vectors (denoted in this paper using the $\searrow$ operator), the arithmetic mean of all elements of any $\mathbf{u} \in US$ may be presented as a re-scaled length of $\mathbf{u} \searrow D_{\mathbf{0}}^{\mathbf{1}}$, i.e., the orthogonal projection of vector $\mathbf{u}$ onto diagonal $D_{\mathbf{0}}^{\mathbf{1}}$. Since the diagonal contains vector $\mathbf{1}$, this simply coincides with the orthogonal projection of $\mathbf{u}$ onto $\mathbf{1}$. Because the length of the projection of any vector $\mathbf{x}$ onto any non-zero vector $\mathbf{y} \neq \mathbf{0}$ is given by $\left|\frac{\mathbf{x} \cdot \mathbf{y}^T}{\|\mathbf{y}\|}\right|$, then (thanks to $\mathbf{1} \neq \mathbf{0}$) the length of the projection of $\mathbf{u} \in US$ onto $\mathbf{1}$ is given by:

$$\left\|\mathbf{u} \searrow D_{\mathbf{0}}^{\mathbf{1}}\right\| \equiv \|\mathbf{u} \searrow \mathbf{1}\| = \left|\frac{\mathbf{u} \cdot \mathbf{1}^T}{\|\mathbf{1}\|}\right| = \left|\frac{\sum_{j=1}^{n} u_j}{\|\mathbf{1}\|}\right| = \left|\frac{sum(\mathbf{u})}{\sqrt{n}}\right|. \tag{1}$$

Since $\mathbf{u} \geq \mathbf{0}$, the absolute value function may be omitted from $\left|\frac{sum(\mathbf{u})}{\sqrt{n}}\right|$, producing $\frac{sum(\mathbf{u})}{\sqrt{n}}$. Finally, $\left\|\mathbf{u} \searrow D_{\mathbf{0}}^{\mathbf{1}}\right\| = \frac{sum(\mathbf{u})}{\sqrt{n}} = \frac{sum(\mathbf{u})}{n} \cdot \sqrt{n} = mean(\mathbf{u}) \cdot \sqrt{n}$, i.e., mean of all elements of $\mathbf{u}$ re-scaled by $\sqrt{n}$.

Similarly, the orthogonal projections of vectors help explain what will be referred to as the *orthogonal distance* $\mathfrak{D}$ between vectors, i.e., the distance in the space orthogonal to $D_{\mathbf{0}}^{\mathbf{1}}$ (Figure 1A). Because the dimensionality of $D_{\mathbf{0}}^{\mathbf{1}}$ equals 1, then the dimensionality of the space that is orthogonal to $D_{\mathbf{0}}^{\mathbf{1}}$ equals $n - 1$. Given vectors $\mathbf{u}_a, \mathbf{u}_b \in US$, the orthogonal distance, denoted by $\mathfrak{D}$, is given by:

$$\begin{aligned}
\mathfrak{D}(\mathbf{u}_a, \mathbf{u}_b) &= \left\|\mathbf{u}_a - mean(\mathbf{u}_a) \cdot \mathbf{1} - (\mathbf{u}_b - mean(\mathbf{u}_b) \cdot \mathbf{1})\right\| \\
&= \left\|(\mathbf{u}_a - \mathbf{u}_b) - (mean(\mathbf{u}_a) \cdot \mathbf{1} - mean(\mathbf{u}_b) \cdot \mathbf{1})\right\| \\
&= \left\|(\mathbf{u}_a - \mathbf{u}_b) - (mean(\mathbf{u}_a) - mean(\mathbf{u}_b)) \cdot \mathbf{1}\right\|.
\end{aligned} \tag{2}$$

Notice that when $mean(\mathbf{u}_a) = mean(\mathbf{u}_b)$, then $(mean(\mathbf{u}_a) - mean(\mathbf{u}_b)) \cdot \mathbf{1} = 0 \cdot \mathbf{1} = \mathbf{0}$, so $\mathfrak{D}(\mathbf{u}_a, \mathbf{u}_b)$ becomes $D(\mathbf{u}_a, \mathbf{u}_b) = \|\mathbf{u}_a - \mathbf{u}_b\|$, i.e. regular Euclidean distance between vectors $\mathbf{u}_a$ and $\mathbf{u}_b$. On the other hand, $\mathfrak{D}(\mathbf{u}_a, \mathbf{u}_b)$ is formally a pseudo-distance, which means that $\mathbf{u}_a = \mathbf{u}_b$ implies $\mathfrak{D}(\mathbf{u}_a, \mathbf{u}_b) = 0$, but $\mathfrak{D}(\mathbf{u}_a, \mathbf{u}_b) = 0$ does not imply $\mathbf{u}_a = \mathbf{u}_b$, as there exist $\mathbf{u}_a \neq \mathbf{u}_b$ such that $\mathfrak{D}(\mathbf{u}_a, \mathbf{u}_b) = 0$ (Figure 1B illustrates both cases).

3 The decision modeling paradigms

Following [Roy, 1996], the most popular decision modeling paradigms are:

- the function-based paradigm ($\mathbb{FUN}$) and
- the relation-based paradigm ($\mathbb{REL}$).

Shortly, these paradigms may be viewed as platforms for defining a family of preference-modeling relations, i.e., *preference* P and *indifference* I. It is crucial that relations P and I are disjoint. The sum $P \cup I$ thus constitutes the ultimate *ordering relation* $\mathcal{O}$. Therefore, the final ranking is the pair $\langle \mathbb{A}, \mathcal{O} \rangle$, where $\mathbb{A}$ is the set of alternatives, while $\mathcal{O}$ is the ordering relation. Pairs of alternatives entering $\mathcal{O}$ in any form will be referred to as *comparable*. Contrastingly, pairs of alternatives not entering $\mathcal{O}$ will be referred to as *incomparable*. Formally, such incomparable pairs will be assumed to enter what will be referred to as the *relation of incomparability*, denoted as R. By this definition, R will be disjoint from $\mathcal{O}$.

Given a binary relation $Rel \subseteq X \times X$, we denote its complement by $Rel^C = (X \times X) \setminus Rel$ and its inverse by $Rel^{-1} = \{(y, x) \,|\, (x, y) \in Rel\}$. With $id = \{(x, x) \,|\, x \in X\}$, Rel is *total* when $Rel \cup Rel^{-1} \cup id = X \times X$ and *partial* otherwise. Of the standard properties of binary relations [Roy, 1996], three combinations recur in this paper: a *weak order* (reflexive, anti-symmetric, transitive), an *equivalence* (reflexive, symmetric, transitive) used to model indifference, and an irreflexive, symmetric relation used to model incomparability.

3.1 Function-based paradigm

The general procedure implementing the function-based paradigm $\mathbb{FUN}$ is carried out in two phases. In phase one, alternatives are *processed individually*. Each alternative $a \in \mathbb{A}$

is subjected in this phase to a function in order to be assigned a real value. In this paper, we assume the following $F : \mathbb{A} \to [0,1]$, further referred to as the *aggregated utility function*. In phase two, the values of the aggregated utility function are used to define preference-modeling relations in which alternatives are processed in pairs. Given $a, b \in \mathbb{A}$, function-based relations of preference $P_{\mathbb{FUN}}$ and indifference $I_{\mathbb{FUN}}$ are defined as follows:

$$\begin{aligned} P_{\mathbb{FUN}} &= \{(a,b) \mid F(a) > F(b), \text{ where } a,b \in \mathbb{A}\}, \\ I_{\mathbb{FUN}} &= \{(a,b) \mid F(a) = F(b), \text{ where } a,b \in \mathbb{A}\}. \end{aligned} \quad (3)$$

First of all, notice that $P_{\mathbb{FUN}}$ and $I_{\mathbb{FUN}}$ are actually independent of the particular counterdomain of the aggregated utility function F. Next, notice that $P_{\mathbb{FUN}}$ and $I_{\mathbb{FUN}}$ are guaranteed to be disjoint. Additionally, while $P_{\mathbb{FUN}}$ is guaranteed to be irreflexive, asymmetric and transitive, $I_{\mathbb{FUN}}$ is guaranteed to be reflexive, symmetric and transitive. All these properties are determined by the definitions of $P_{\mathbb{FUN}}$ and $I_{\mathbb{FUN}}$, but basically independent of how function F is defined.

Now, using $P_{\mathbb{FUN}}$ and $I_{\mathbb{FUN}}$ we define $\mathcal{O}_{\mathbb{FUN}} = P_{\mathbb{FUN}} \cup I_{\mathbb{FUN}}$, the ordering relation, which in this case is a total pre-order. Finally, we define $R_{\mathbb{FUN}} = (\mathcal{O}_{\mathbb{FUN}} \cup \mathcal{O}_{\mathbb{FUN}}^{-1})^C$. Because $\mathcal{O}_{\mathbb{FUN}}$ is by definition total, then $R_{\mathbb{FUN}} = \varnothing$. Incomparability is thus unavailable in $\mathbb{FUN}$.

3.2 Relation-based paradigm

The general procedure implementing the relation-based paradigm $\mathbb{REL}$ is also carried out in two phases. In phase one, alternatives are *processed in pairs*. Each pair of alternatives $a, b \in \mathbb{A}$ is subjected to a valued binary relation, which is a function assigning real values to pairs of alternatives. In this paper, we assume the following $S : \mathbb{A} \times \mathbb{A} \to [0,1]$, further referred to as the *relation of outranking*. In particular, S is assumed to ensure $S(a,a) = 1$ for every $a \in \mathbb{A}$.

In phase two, the values assigned by S to the pairs of alternatives are used to define preference-modeling relations. Given $a, b \in \mathbb{A}$, relations of preference $P^{\lambda}_{\mathbb{REL}}$ and indifference $I^{\lambda}_{\mathbb{REL}}$ are defined as follows:

$$\begin{aligned} P^{\lambda}_{\mathbb{REL}} &= \{(a,b) \mid S(a,b) \geq \lambda \wedge S(b,a) < \lambda, \text{where } a,b \in \mathbb{A}\}, \\ I^{\lambda}_{\mathbb{REL}} &= \{(a,b) \mid S(a,b) \geq \lambda \wedge S(b,a) \geq \lambda, \text{where } a,b \in \mathbb{A}\}, \end{aligned} \quad (4)$$

where $\lambda \in [0,1]$ is a parameter specified by the DM. First of all, notice that $P_{\mathbb{REL}}$ and $I_{\mathbb{REL}}$ are actually independent of the particular counterdomain of the valued relation S. Again, $P^{\lambda}_{\mathbb{REL}}$ and $I^{\lambda}_{\mathbb{REL}}$ are disjoint. This is determined by the definitions of $P^{\lambda}_{\mathbb{REL}}$ and $I^{\lambda}_{\mathbb{REL}}$, but basically independent of how function S is defined. Additionally, a specific property of S, namely: $S(a,a) = 1$ for every $a \in A$, guarantees that while $P^{\lambda}_{\mathbb{REL}}$ is irreflexive and asymmetric, $I^{\lambda}_{\mathbb{REL}}$ is reflexive and symmetric. Notice that the lack of transitivity may be treated as a weakness of $\mathbb{REL}$.

Now, using $P^{\lambda}_{\mathbb{REL}}$ and $I^{\lambda}_{\mathbb{REL}}$, we define $\mathcal{O}^{\lambda}_{\mathbb{REL}} = P^{\lambda}_{\mathbb{REL}} \cup I^{\lambda}_{\mathbb{REL}}$, which in this case is partial. Finally, we define $R^{\lambda}_{\mathbb{REL}} = (\mathcal{O}^{\lambda}_{\mathbb{REL}} \cup (\mathcal{O}^{\lambda}_{\mathbb{REL}})^{-1})^C$. When $\mathcal{O}^{\lambda}_{\mathbb{REL}}$ is not total, then $R^{\lambda}_{\mathbb{REL}} \neq \varnothing$. Incomparability is thus naturally available in $\mathbb{REL}$.

4 Dominance-inspired extensions to the function-based paradigm

Dominance is a relation commonly used in various MCDA approaches. Despite its limitations for direct use as a preference modeling relation—notably its tendency towards an excessively poor indifference relation and an excessively rich incomparability relation—dominance shall constitute the main inspiration for extending the functional paradigm (Eq. 3) with incomparability. To this end, we introduce the notion of *commonality*, grounded in the dominance relation. Examining commonality leads to the key insight that displacement perpendicular to the diagonal $D_{\mathbf{0}}^{\mathbf{1}}$ is the most intuitive way to identify well-justified instances of incomparability. Building on this, we will finally show how a prototypical example of a function-based model can incorporate a meaningful incomparability relation through the application of orthogonal distance.

4.1 Dominance-based family of preference-modeling relations

One must bear in mind, that there are many different variants of definitions of dominance in the literature. In this paper, *dominance* (more formally: *weak dominance*) is defined with the help of US as follows:

$$Dom_{\geq} = \{(a,b) \mid \mathbf{u}_a \geq \mathbf{u}_b \text{ where } a,b \in \mathbb{A}\}. \quad (5)$$

Recall that given an alternative $a \in \mathbb{A}$, $\mathbf{u}_a \in US$ represents this alternative in US (similarly with $b \in \mathbb{A}$, $\mathbf{u}_b \in US$, etc.).

Because $Dom_{\geq}$ is a partial weak order (a Pareto order), the family of preference-modeling relations may be formulated as:

$$\begin{aligned} P_{Dom} &= Dom_{\geq} \setminus Dom_{\geq}^{-1}, \\ I_{Dom} &= Dom_{\geq} \cap Dom_{\geq}^{-1}. \end{aligned} \quad (6)$$

These definitions imply that P_{Dom} and I_{Dom} are pairwise disjoint, with P_{Dom} a partial strict order and I_{Dom} an equivalence. Now, using P_{Dom} and I_{Dom}, we define an ordering $\mathcal{O}_{Dom}$ as $\mathcal{O}_{Dom} = P_{Dom} \cup I_{Dom}$. If $n = 1$, then $\mathcal{O}_{Dom}$ is a total weak order (and thus a total pre-order). In the more general case of $n > 1$, $\mathcal{O}_{Dom}$ is a partial weak order (thus a partial pre-order).

At this point, incomparability may be defined as:

$$R_{Dom} = (\mathcal{O}_{Dom} \cup \mathcal{O}_{Dom}^{-1})^C. \quad (7)$$

Of course, for $n = 1$, $R_{Dom} = \varnothing$. By definition, $\mathcal{O}_{Dom} \cup R_{Dom}$ is total for every $n \geq 1$.

Notice that the family of preference-modeling relations defined in this way is inherently of preference-shallow nature, as nothing enters its definitions apart from the absolutely minimal amount of preferential information, i.e., information on the type of the criteria, necessary for establishing dominance. Unfortunately, such formulations may obviously fail to express any actual preferences. Additionally, the family does contain what may be considered a flaw, which is that in most practical cases P_{Dom} and especially I_{Dom} will be excessively poor, while R_{Dom} will be excessively rich. The excessively poor cardinality of I_{Dom} is fairly obvious as I_{Dom} consists only of pairs (a,b) characterized by representations $\mathbf{u}_a \in US$ and

$\mathbf{u}_b \in US$ satisfying $\mathbf{u}_a = \mathbf{u}_b$, which by definition includes pairs (a, a), and very often nothing else. On the other hand, the poor cardinality of P_{Dom} and the excessive cardinality of R_{Dom} may be less obvious and will be best explained with the following probability calculations.

Consider alternative a and its representation in the form of $\mathbf{u}_a = 0.5 \cdot \mathbf{1}$ (this vector is chosen in a specific way: it occupies the very centre of the US, but this fact does not decrease the generality of the considerations). In the case of 2-dimensional US ($n = 2$) considered here, $\mathbf{u}_a = 0.5 \cdot \mathbf{1} = [0.5, 0.5]$. Figure 2A depicts the vector and its four cones (lower-left, lower-right, upper-right, upper-left), necessarily truncated to US. These include the cones of main interest, i.e. the two dominance cones: the cone of vectors dominating $\mathbf{u}_a$ (upper-right, pink rectangle) and the cone of vectors dominated by $\mathbf{u}_a$ (lower-left, blue rectangle). Similarly, consider alternative b, for which Figure 2B depicts its representative $\mathbf{u}_b \in US$, boundaries of its four cones, including the two truncated dominance cones: the cone of vectors dominating $\mathbf{u}_b$ (pink rectangle) and the cone of vectors dominated by $\mathbf{u}_b$ (blue rectangle).

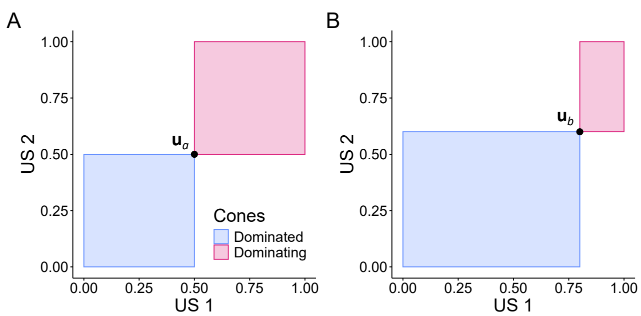


Figure 2: Two alternatives represented as vectors of US: (A) $\mathbf{u}_a = [0.5, 0.5]$, (B) $\mathbf{u}_b = [0.8, 0.6]$. The pink and blue colored rectangles represent their truncated dominating and dominated cones, respectively.

Pair (a, b) enters I_{Dom} when $\mathbf{u}_b = \mathbf{u}_a$, enters P_{Dom} (in one direction) when $\mathbf{u}_b$ lies in either truncated dominance cone of $\mathbf{u}_a$ (e.g. $\mathbf{u}_b = [0.8, 0.6]$ or $[0.6, 0.8]$; Figure 2), and enters R_{Dom} otherwise (e.g. $[0.8, 0.2]$ or $[0.2, 0.8]$; Figure 2).

The probabilities of preference and incomparability can be quantified. Given n-dimensional US with uniform density and $\mathbf{u}_a = 0.5 \cdot \mathbf{1}$, the two truncated dominance cones of $\mathbf{u}_a$ each occupy $\frac{1}{2^n}$ of the volume of US, so the probability that a randomly drawn $\mathbf{u}_b \in US$ enters P_{Dom}, I_{Dom}, or R_{Dom} with $\mathbf{u}_a$ is $\frac{1}{2^{n-1}}$, 0, and $1 - \frac{1}{2^{n-1}}$, respectively. The formulae are more involved for non-centered $\mathbf{u}_a$, but the qualitative conclusion is invariant: as n grows, P_{Dom} rapidly becomes vanishingly small while R_{Dom} approaches totality, confirming the impractical richness of R_{Dom} in realistic multi-criteria settings.

Interestingly, with incomparability R_{Dom} being excessively rich in general, it is expected that individual cases of incomparability may be more justified or less justified. Cases when proportions of the elements in the two vectors $\mathbf{u}_a$ and $\mathbf{u}_b$ are mostly different seem more justified; on the other hand, cases when proportions of the elements of the two vectors are the same or similar are less justified. Identifying those less justified cases may well be used for 'slimming down' the originally rich incomparability relation. Such a modification is necessary for the relation to be of practical usage.

4.2 The notion of commonality

The process of 'slimming down' the incomparability relation should start with some well-founded quantification of the content of this relation. Below we demonstrate how this quantification can be established on a dominance-based line of reasoning.

Consider what may be referred to as the *coefficient of dominance commonality* $\mathfrak{C}$ (shortly: commonality coefficient or commonality) of vectors $\mathbf{u}_a = [a_1, a_2, ..., a_n] \in US$ and $\mathbf{u}_b = [b_1, b_2, ..., b_n] \in US$, defined as the sum of the area of the common parts of their truncated dominating cones and the area of the common parts of their truncated dominated cones (Figure 3). The coefficient thus expresses the commonality of both of their truncated dominance cones and is defined as:

$$\mathfrak{C}(\mathbf{u}_a, \mathbf{u}_b) = \prod_{j=1}^{n} min(a_j, b_j) + \prod_{j=1}^{n} (1 - max(a_j, b_j)). \quad (8)$$

Notice that the counterdomain of $\mathfrak{C}(\mathbf{u}_a, \mathbf{u}_b)$ is $[0, 1]$, however, in situations with one argument constant it is $[0, M]$, where $0 \leq M \leq 1$ (M depends on the choice of the constant argument). The maximum (M) is achieved exclusively for $\mathbf{u}_a = \mathbf{u}_b$, while the minimum (0) is achieved for all such $\mathbf{u}_b$ that have nothing in common with $\mathbf{u}_a$ in terms of their truncated dominance cones. Of course the coefficient is by definition symmetric in the sense that $\mathfrak{C}(\mathbf{u}_a, \mathbf{u}_b) = \mathfrak{C}(\mathbf{u}_b, \mathbf{u}_a)$ for all $\mathbf{u}_a, \mathbf{u}_b \in US$.

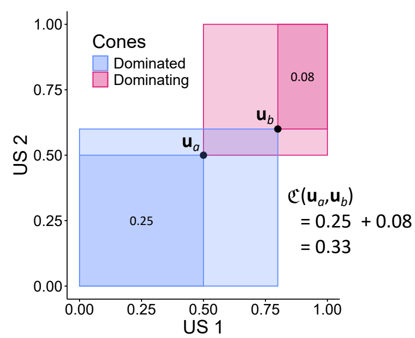


Figure 3: Two vectors of US: $\mathbf{u}_a = [0.5, 0.5]$ and $\mathbf{u}_b = [0.8, 0.6]$ with their dominance cones and common parts of their dominating cones (shown in dark red) and the common parts of their dominated cones (shown in dark blue). Commonality is computed as $\mathfrak{C}(\mathbf{u}_a, \mathbf{u}_b) = 0.5 \cdot 0.5 + (1 - 0.6) \cdot (1 - 0.8) = 0.25 + 0.08 = 0.33$.

Notice that when $\mathbf{u}_a = [0.5, 0.5]$ is constant and $\mathbf{u}_b$ varies, the counterdomain of $\mathfrak{C}(\mathbf{u}_a, \mathbf{u}_b)$ is $[0.0, 0.5]$. This is because for vector $\mathbf{u}_b = \mathbf{u}_a$ the value of the commonality coefficient will be equal $\mathfrak{C}(\mathbf{u}_a, \mathbf{u}_b) = \mathfrak{C}(\mathbf{u}_a, \mathbf{u}_a) = 0.25 + 0.25 = 0.50$, achieving its particular maximum of 0.5. Clearly, high values of the commonality between two vectors mean that the vectors have much in common as far as their dominating and dominated truncated cones are concerned. With $\mathbf{u}_b$ departing from $\mathbf{u}_a$ the value will get smaller, potentially down to 0.

Figure 4A shows values (rendered as color) of the commonality coefficient of each possible vector $\mathbf{u}_b$ of 2-dimensional US with a constant vector $\mathbf{u}_a = [0.50, 0.50]$. As expected, the value achieves its maximum (0.50) for $\mathbf{u}_b = \mathbf{u}_a$ (in such a case alternatives a and b are clearly indifferent in the sense of I_{Dom}), and decreases when $\mathbf{u}_b$ moves away from $\mathbf{u}_a$, potentially down to its minimum (0.00) (in such a case alternatives a and b obviously stop being indifferent and become either preferred or incomparable). However, the following two special cases apply:

- $\mathfrak{C}(\mathbf{u}_a, \mathbf{u}_b)$ never drops below 0.25 when $\mathbf{u}_b$ departs from $\mathbf{u}_a$ in a direction parallel to $D^{\mathbf{1}}_{\mathbf{0}}$. In this case, $mean(\mathbf{u}_b) \neq mean(\mathbf{u}_a)$ and alternatives a and b may thus enter the relation of preference.
- $\mathfrak{C}(\mathbf{u}_a, \mathbf{u}_b)$ drops all the way to 0.00 when $\mathbf{u}_b$ departs from $\mathbf{u}_a$ in a direction perpendicular to $D^{\mathbf{1}}_{\mathbf{0}}$. In this case $mean(\mathbf{u}_b) = mean(\mathbf{u}_a)$ and alternatives a and b may thus enter the relation of incomparability.

In all other cases, i.e. when $\mathbf{u}_b$ departs from $\mathbf{u}_a$ in any intermediate direction, the outcome (regarding the means of these vectors and the new relation of represented alternatives) depends on which constituent of its displacement (parallel or perpendicular) was larger.

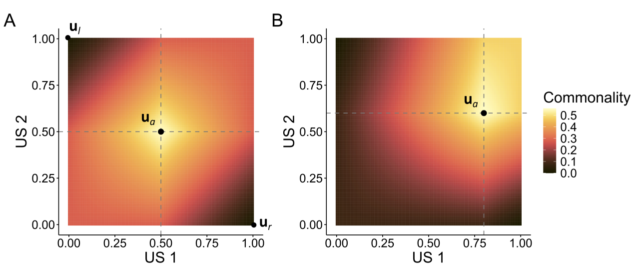


Figure 4: The 2D graph of $\mathfrak{C}(\mathbf{u}_a, \mathbf{u}_b)$: the commonality coefficient of constant $\mathbf{u}_a$ with every $\mathbf{u}_b \in US$. The resulting commonality value is rendered using color. (A) Constant $\mathbf{u}_a = [0.5, 0.5]$. Vectors $\mathbf{u}_l$ and $\mathbf{u}_r$ represent the special case of lowest possible commonality. (B) Constant $\mathbf{u}_a = [0.8, 0.6]$. Notice that placing $\mathbf{u}_a$ away from the center of US (i.e, [0.5, 0.5]) influenced the $\mathfrak{C}(\mathbf{u}_a, \mathbf{u}_b)$ due to the truncation of the dominance cones to the confines of US.

In particular, while the maximal value of the coefficient is achieved exclusively for one vector, the minimal value (i.e. 0.00) is achieved for two vectors: $\mathbf{u}_b = \mathbf{u}_r$, where $\mathbf{u}_r = [1, 0]$, and $\mathbf{u}_b = \mathbf{u}_l$, where $\mathbf{u}_l = [0, 1]$ (Figure 4A).

The general conclusion is that for an alternative $\mathbf{u}_b$ to be considered 'maximally' incomparable with alternative $\mathbf{u}_a$ in the sense of R_{Dom} it should be maximally displaced from $\mathbf{u}_a$ in a direction perpendicular to $D^{\mathbf{1}}_{\mathbf{0}}$. What is important, this particular form of the displacement modifies the elements of the vector but maintains their sum.

In the example, two cases of such maximal displacement arise: pair $\mathbf{u}_a$ with $\mathbf{u}_r$ and pair $\mathbf{u}_a$ with $\mathbf{u}_l$, displaced by the radius of US. They are characterized by zero commonality and thus constitute well justified cases of incomparability. Notice that what comes with zero commonality is the fact that the considered pairs of vectors, i.e. $(\mathbf{u}_a, \mathbf{u}_r)$ and $(\mathbf{u}_a, \mathbf{u}_l)$, are characterized by a very different proportion of their elements as compared to $\mathbf{u}_a$: 0 and 1 or 1 and 0 versus 0.5 and 0.5. Simultaneously, zero commonality signifies that the vectors have basically nothing in common as far as their dominating and dominated truncated cones are concerned. Notice that by the nature of perpendicular displacement vectors $\mathbf{u}_r$ and $\mathbf{u}_l$ are characterized by exactly the same mean as $\mathbf{u}_a$, i.e., $mean([0, 1]) = mean([1, 0]) = 0.5$, $mean([0.5, 0.5]) = 0.5$. Note, however, that commonality floors at zero while perpendicular displacement (and the underlying disproportion of vector elements) continues to grow beyond this point, motivating the more direct measure introduced next.

4.3 Extending the function-based paradigm using orthogonal distance

As established in Section 4.2, well-justified incomparability corresponds to displacement perpendicular to $D^{\mathbf{1}}_{\mathbf{0}}$. The commonality coefficient $\mathfrak{C}$ captures this geometrically, but is distorted by truncation when $\mathbf{u}_a$ lies near the boundary of US (Figure 4B). The orthogonal distance $\mathfrak{D}$ (Section 2.3) quantifies the same perpendicular displacement directly and remains stable across US.

We now use $\mathfrak{D}$ to define a family of preferential relations equipped with incomparability in a preference-shallow setting. As an example implementation of this approach, we extend a ranking method in which the arithmetic mean of utilities serves as the aggregated utility function.

Let $a, b \in \mathbb{A}$ be alternatives and $\mathbf{u}_a, \mathbf{u}_b \in US$ their representations. Consider an instance of $\mathbb{FUN}$ with a family of preferential relations $\mathcal{O}_m = P_m \cup I_m$ as follows (notice the usage of $mean(\mathbf{u})$ in the role of the aggregated utility function):

$$\begin{aligned} P_m &= \{(a, b) \mid mean(\mathbf{u}_a) > mean(\mathbf{u}_b)\}, \\ I_m &= \{(a, b) \mid mean(\mathbf{u}_a) = mean(\mathbf{u}_b)\}. \end{aligned} \tag{9}$$

In such a case, $\mathcal{O}_m$ is a total pre-order. As already stated, because $\mathcal{O}_m$ is by definition total, then $R_m = (\mathcal{O}_m \cup \mathcal{O}_m^{-1})^C = \varnothing$, i.e., the family does not admit incomparability.

Now, consider $\mathcal{O}_{\mathfrak{D}} = P_{\mathfrak{D}} \cup I_{\mathfrak{D}}$, an adaptation of $\mathcal{O}_m$, in which $\mathfrak{D}(\mathbf{u}_a, \mathbf{u}_b)$, i.e. the orthogonal distance (Equation 2) between $\mathbf{u}_a$ and $\mathbf{u}_b$ is used as a measure of 'perpendicular displacement'. The new family is:

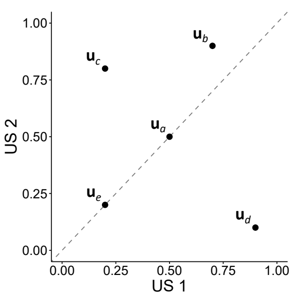
$$\begin{aligned} P_{\mathfrak{D}} &= \{(a, b) \mid mean(\mathbf{u}_a) > mean(\mathbf{u}_b)\}, \\ I_{\mathfrak{D}} &= \{(a, b) \mid mean(\mathbf{u}_a) = mean(\mathbf{u}_b) \wedge \mathfrak{D}(\mathbf{u}_a, \mathbf{u}_b) \leq \sigma\}, \end{aligned} \tag{10}$$

where σ (the orthogonal distance threshold), is a parameter specified by the DM. Because $\sqrt{n}$ is the diameter of US, it constitutes the upper limit for any distance between elements of US, including $\mathfrak{D}(\mathbf{u}_a, \mathbf{u}_b)$. For this reason, σ should stay in $[0, \sqrt{n}]$, with $\sigma = \sqrt{n}$ eliminating all instances of incomparability. However, we very strongly suggest using $\sigma \in (\sqrt{n}/2, \sqrt{n}]$, because only orthogonal distances higher than $\sqrt{n}/2$ ensure that the different criterion values 'oscillate' enough around their respective 'midpoints' (some being better, others worse), which lends those cases the natural interpretation of incomparability.

$\mathcal{O}_{\mathfrak{D}}$ is partial when $R_{\mathfrak{D}} = (\mathcal{O}_{\mathfrak{D}} \cup \mathcal{O}_{\mathfrak{D}}^{-1})^C \neq \varnothing$ and total otherwise; in either case $\mathcal{O}_{\mathfrak{D}} \cup R_{\mathfrak{D}}$ is a total pre-order. Although $\mathcal{O}_{\mathfrak{D}}$ thus acquires the incomparability characteristic of $\mathbb{REL}$ families, it retains a 'lower-cost' profile than typical $\mathbb{REL}$ relations: $P_{\mathfrak{D}}$ remains transitive, and although $I_{\mathfrak{D}}$ loses transitivity, $I_{\mathfrak{D}} \cup R_{\mathfrak{D}}$ stays transitive. What is also important, this particular way of defining $R_{\mathfrak{D}}$ makes it invariably irreflexive and symmetric (thus following the expectations).

4.4 Illustrative example and numerical evaluation

To best illustrate the proposed $\mathcal{O}_{\mathfrak{D}}$ consider representations $\mathbf{u}_a..\mathbf{u}_e \in US$ $(n = 2)$ of five alternatives a, b, c, d, e, as shown in Figure 5. We will use a σ equal to 1.2 of the radius, producing $\sigma = 1.2 \cdot \sqrt{n}/2 = 1.2 \cdot \sqrt{2}/2 = 0.85$.



alternative	$\mathbf{u}_i$	$mean(\mathbf{u}_i)$
a	$[0.50, 0.50]$	0.5
b	$[0.70, 0.90]$	0.8
c	$[0.20, 0.80]$	0.5
d	$[0.90, 0.10]$	0.5
e	$[0.20, 0.20]$	0.2

Figure 5: The five example vectors $\mathbf{u}_i$ for $i \in \{a, b, c, d, e\}$ in 2-dimensional US (the dashed line depicts $D_{\mathbf{0}}^{\mathbf{1}}$).

Notice that the equal mean of vectors $\mathbf{u}_a$, $\mathbf{u}_c$ and $\mathbf{u}_d$ is manifested by the fact that their orthogonal projections onto $D_{\mathbf{0}}^{\mathbf{1}}$ are all equal to $\mathbf{u}_a$ and therefore identical: $\mathbf{u}_a \searrow D_{\mathbf{0}}^{\mathbf{1}} = \mathbf{u}_c \searrow D_{\mathbf{0}}^{\mathbf{1}} = \mathbf{u}_d \searrow D_{\mathbf{0}}^{\mathbf{1}}$.

Table 1 additionally specifies the orthogonal distances of the 25 pairs of vectors. It should be stressed that the distances presented there are not regular distances. Instead, they are distances calculated in the space that is orthogonal to $D_{\mathbf{0}}^{\mathbf{1}}$; in the case of $n = 2$ they are simply distances calculated along an $n - 1 = 2 - 1 = 1$ dimensional line, namely one that is perpendicular to $D_{\mathbf{0}}^{\mathbf{1}}$. In consequence, values of the orthogonal distance may be equal to or smaller than the regular distance. For example, vectors $\mathbf{u}_a$ and $\mathbf{u}_c$ have identical means, so $\mathfrak{D}(\mathbf{u}_a, \mathbf{u}_c)$ is equal to the regular distance between them. On the other hand, vectors $\mathbf{u}_a$ and $\mathbf{u}_e$ differ noticeably in terms of their means, but happen to be positioned so that $\mathfrak{D}(\mathbf{u}_a, \mathbf{u}_e)$ is zero, even though the regular distance between $\mathbf{u}_a$ and $\mathbf{u}_e$ is clearly non-zero.

Table 1: Orthogonal distances $\mathfrak{D}$ between pairs of example vectors.

$\times$	$\mathbf{u}_a$	$\mathbf{u}_b$	$\mathbf{u}_c$	$\mathbf{u}_d$	$\mathbf{u}_e$
$\mathbf{u}_a$	0.00	0.14	0.42	0.57	0.00
$\mathbf{u}_b$	0.14	0.00	0.28	0.71	0.14
$\mathbf{u}_c$	0.42	0.28	0.00	0.99	0.42
$\mathbf{u}_d$	0.57	0.71	0.99	0.00	0.57
$\mathbf{u}_e$	0.00	0.14	0.42	0.57	0.00

We now compare the function-based relations of $\mathcal{O}_m$ with its commonality-inspired extension $\mathcal{O}_{\mathfrak{D}}$. According to the original $\mathcal{O}_m$ (Equation 9), $P_m = \{(b, a), (b, c), (b, d), (b, e), (a, e), (c, e), (d, e)\}$ and $I_m = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, c), (a, d), (c, a), (c, d), (d, a), (d, c)\}$, therefore $\mathcal{O}_m$ consists of 18 pairs. The inverse $\mathcal{O}_m^{-1}$ contributes 7 further pairs $\{\ (a, b), (c, b), (d, b), (e, b), (e, a), (e, c), (e, d)\ \}$, thus $\mathcal{O}_m \cup \mathcal{O}_m^{-1}$ exhausts all 25 possible pairs and $R_m = (\mathcal{O}_m \cup \mathcal{O}_m^{-1})^C = \varnothing$ admits no incomparability.

The resulting $\mathcal{O}_m$-based ranking places alternative b at the highest (best) level, alternatives a, c, and d at the middle level (all regarded as indifferent and worse than b), and alternative e at the lowest level as the least preferred.

For the adapted $\mathcal{O}_{\mathfrak{D}}$ (Equation 10), $P_{\mathfrak{D}} = P_m$ by definition, while the indifference relation now uses the σ threshold. With $\sigma = 0.85$ in this example, Table 1 shows that pairs (c, d) and (d, c) have $\mathfrak{D} = 0.99 > \sigma$, so $I_{\mathfrak{D}}$ drops these two pairs from I_m, giving $I_{\mathfrak{D}} = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, c), (a, d), (c, a), (d, a)\}$. The same set operations then yield $R_{\mathfrak{D}} = (\mathcal{O}_{\mathfrak{D}} \cup \mathcal{O}_{\mathfrak{D}}^{-1})^C = \{(c, d), (d, c)\}$ — the family $\mathcal{O}_{\mathfrak{D}}$ admits incomparability.

Table 2 presents both rankings side by side; the only difference is in the second level, where $\mathcal{O}_{\mathfrak{D}}$ exposes the incomparability of (c, d). Notice the three alternatives at this level, represented by vectors: $\mathbf{u}_a = [0.50, 0.50]$, $\mathbf{u}_c = [0.20, 0.80]$, $\mathbf{u}_d = [0.90, 0.10]$ (Figure 5). Among these three alternatives, pairs of indifferent alternatives (a, c) and (a, d) are characterized by fairly similar proportions of the corresponding vector elements. Contrastingly, pair (c, d) is characterized by utterly different proportions of the corresponding vector elements, and this is what best justifies treating alternatives of this pair as incomparable.

Table 2: Rankings of the five considered alternatives under $\mathcal{O}_m$ (function-based) and $\mathcal{O}_{\mathfrak{D}}$ (its orthogonal-distance extension). Consecutive rows present consecutive ranking levels, with each level enumerating the alternatives indifferent at that level and listing the pairs incomparable at that level. The two families differ only in $R_{\mathfrak{D}}^2$.

level	alternatives	R_m	$R_{\mathfrak{D}}$
1	b	$R_m^1 = \varnothing$	$R_{\mathfrak{D}}^1 = \varnothing$
2	a, c, d	$R_m^2 = \varnothing$	$R_{\mathfrak{D}}^2 = \{(c, d), (d, c)\}$
3	e	$R_m^3 = \varnothing$	$R_{\mathfrak{D}}^3 = \varnothing$

We also evaluated the proposed dominance-inspired extensions on the QS World University Rankings 2026 dataset [Quacquarelli Symonds, 2025], comprising 1,501 universities assessed across 10 criteria. Five approaches were compared: Pure Dominance as the baseline, PROMETHEE I and ELECTRE III representing the relational paradigm, and SAW$_{\mathfrak{D}}$ and TOPSIS$_{\mathfrak{D}}$ as the proposed function-based paradigm extended with incomparability. All criteria were normalized into $US = [0, 1]^{10}$, and we grid-searched each method's thresholds. Out of 1,125,750 unique pairs of universities, Pure Dominance yielded a constant 961,041 incomparable pairs (85.37%), confirming that R_{Dom} becomes excessively rich in high-dimensional spaces and impractical for real-world rankings. The relational methods reduced this substantially but still averaged 84,082 (7.47%) incomparable pairs for ELECTRE III and 66,210 (5.88%) for

PROMETHEE I. By contrast, SAW$_{\mathfrak{D}}$ and TOPSIS$_{\mathfrak{D}}$ averaged only 1,478 (0.13%) and 1,922 (0.17%) pairs, respectively, restricting incomparability to pairs that are both globally indifferent and highly disproportionate in their criteria profiles. This confirms that the functional paradigm can model incomparability without requiring complex parameter elicitation.

5 Conclusions and future work

We have proposed an extension of the function-based paradigm that admits incomparability while remaining preference-shallow. The approach is inspired by the objective dominance relation: we introduced the commonality coefficient as a dominance-grounded measure of cone overlap, identified its limitations, and showed that orthogonal distance $\mathfrak{D}$ perpendicular to D_0^1 provides a more direct and stable proxy for dominance-inspired incomparability. The resulting family $\mathcal{O}_{\mathfrak{D}}$ enriches function-based ranking methods with well-justified incomparability at the cost of a single threshold σ, while retaining transitivity of $P_{\mathfrak{D}}$ and $I_{\mathfrak{D}} \cup R_{\mathfrak{D}}$.

Our extension is one of several possible solutions. Natural directions for future work include relaxing $I_{\mathfrak{D}}$ from an equivalence to a tolerance, and incorporating criterion weights to move the construction toward preference-deeper but more practically applicable territory.

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